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A principle of invariant imbedding with memory

Eugene d'Eon[1]

[1] *ejdeon@gmail.com, The Jig Lab, Wellington NZ*

1 Non-Classical Transport Theory with Memory

It is of considerable interest to expand the applicability of transport theory to wider domains by considering non-classical transport models. Non-exponential random flights are one such example [1,2]. Enhanced backscattering (due to the *opposition* or *hotspot effect*) is not exhibited by classical transport theory. This is not a wave/interference phenomena—the effects are found in purely geometrical optics simulations of discrete random media [3,4]. We investigate the possibility of modifying the method of invariant imbedding to incorporate a notion of memory. We perform this analysis in the rod model where the effects of memory are exhibited more strongly than in any other possible geometry. This is also a domain where explicit realizations of the entire rod medium are practical to both randomly sample and to efficiently compute transport solutions within. This is critical to the validation of our transport solution, which we show to be in good agreement with this ground-truth simulation. Being able to predict the effects of discrete random media without the burden of heavy random media realizations and tracing [3] is our primary motivation.

Classical principle of invariant imbedding: We consider time-independent monoenergetic transport in the rod model [5]. Let $R(x)$ be the reflectance from a homogeneous rod of thickness x with interaction coefficient Σ_t and scattering kernel $\{F, B\}$. We construct a differential equation for R by considering how R changes as a rod of diminishing thickness Δ is added to a rod of thickness x (Figure 1a). A glass plates theory analysis leads to the new reflectance of the thicker rod in terms of the original rod reflectance, $R(x)$. The probability of one scatterer existing in the small rod segment of length Δ is

$$p_1 = \Sigma_t \Delta + o(\Delta). \tag{1}$$

The probability of multiple scattering events within the segment of thickness Δ involves terms of order at least $(\Sigma_t \Delta)^2$ so it is covered by $o(\Delta)$. Computing R and T for the small segment gives [5] a differential equation for the reflectance of a rod of thickness x:

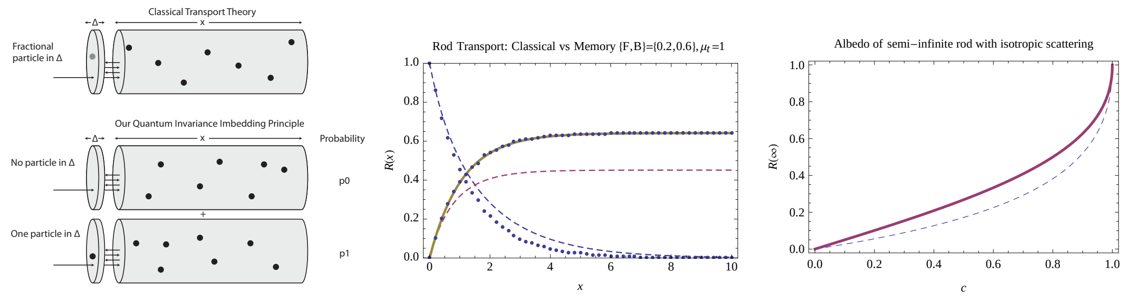
$$\frac{dR(x)}{dx} = B\Sigma_t R(x)^2 + B\Sigma_t + 2F\Sigma_t R(x) - 2\Sigma_t R(x). \tag{2}$$

The Albedo problem for semi-infinite rod with isotropic scattering ($F = B$, single-scattering albedo of $c = F + B$)

$$\lim_{x\to\infty} R(x) = \frac{2\left(-\frac{c}{2} - \sqrt{1-c} + 1\right)}{c}. \tag{3}$$

A Quantum Invariant Imbedding Principle: We now repeat the invariant imbedding analysis for the reflectance of a finite rod, but require that the occurence of a single scatterer in the thin rod of length Δ influence all orders of interreflections between the two rods. The invariant imbedding analysis is now different (see Figure 1a): we have two glass plates calculations. One, with $R_1 = 0, T_1 = 1$ with probability $p_0 = 1 - (\Sigma_t \Delta + o(\Delta))$ and one with $R_1 = B, T_1 = F$ with probability $p_1 = \Sigma_t \Delta + o(\Delta)$. $R_2 = R(x)$ in both, leading to

$$\frac{dR(x)}{dx} = \left(B\,\Sigma_t - \Sigma_t\,R(x) + \frac{F^2\,\Sigma_t\,R(x)}{1 - B\,R(x)}\right). \tag{4}$$



(a) (b) (c)

Figure 1: (b) Reflectance and transmittance as a function of rod length x. Dots: Memory Monte Carlo, thick curve: our new memory solution, dashed: classical transport theory (c) Albedo of a semi-infinite rod with isotropic scattering (single-scattering albedo of c) for classical transport theory (dashed) vs. memory transport theory (thick).

This can be solved using standard ODE techniques but involves the inverse of a function with no simple analytic form. However, it is easily inverted numerically. The solution to the albedo problem for a semi-infinite rod with isotropic scattering (for comparison to Equation 3), however, is quite simple,

$$\lim_{x\to\infty} R(x) = \frac{1-\sqrt{1-c^2}}{c}. \tag{5}$$

Comparison of the two transport models: We test the predictions of our new transport derivations using a Monte Carlo estimator consistent with the thought process used to derive transport with memory. To form a Monte carlo estimator for the memory model of a rod of length x, we start at one end of the rod and repeatedly sample the distance to the next scattering location, but always in the forward direction. We sample each next event using $x_i = -\frac{\log 1-\xi_i}{\Sigma_t}$ (where ξ_i are uniform random numbers in $[0,1)$). We stop when $\sum_i x_i > x$. This gives an estimate for an integer number of scatters in the rod, $i-1$, from which we can easily compute a glass plates $R(x)$ and $T(x)$. Figure 1 b and c compares both forms of transport theory including the Monte Carlo version of transport with memory, showing good agreement for the $R(x)$ prediction using the quantum invariant imbedding principle. The albedo problem solution shows significant differences between classical and memory transport.

References

[1] Larsen, E.W. and Vasques, R., "A generalized linear Boltzmann equation for non-classical particle transport," *Journal of Quantitative Spectroscopy and Radiative Transfer*, vol. 112 (4), pp. 619–631 (2011).

[2] Barthelemy, Pierre and Bertolotti, Jacopo and Wiersma, Diederik S, "A Lévy flight for light," *Nature*, vol. 453 (7194), pp. 495–498 (2008).

[3] Moon, J.T. and Walter, B. and Marschner, S.R., "Rendering discrete random media using precomputed scattering solutions," *Rendering Techniques*, vol. 1, pp. 231–242 (2007).

[4] Olson, Gordon L., "Chord length distributions between hard disks and spheres in regular, semi-regular, and quasi-random structures," *Annals of Nuclear Energy*, vol. 35 (11), pp. 2150–2155 (2008).

[5] Wing, G.M., *An introduction to transport theory*, Wiley, (1962).