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The Albedo Problem in Nonexponential Radiative Transfer

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We derive exact solutions of the generalized albedo problem of isotropic scattering in a half space in $\mathbb{R}^d$ with smooth vacuum boundary under monodirectional and uniform diffuse illumination conditions. We consider general dimension $d \geq 1$ and nonclassical transport $[1, 2]$ with a general, nonexponential free-path distribution $p_c(s)$ between collisions and nonstochastic phase function P and single-scattering albedo $0 < c \leq 1$, where absorption is restricted to the collision sites. Photons entering the medium from the boundary draw their initial free-path lengths from distribution $p_u(s)$. The two attenuation laws are $X_c(s) = 1 - \int_0^s p_c(s')ds'$ between collisions and $X_u(s) = 1 - \int_0^s p_u(s')ds'$ from the boundary.

We derive the Green's function for the half space and the general law of diffuse reflection (BRDF) and diffuse albedo are also attained, provided the Fourier and inverse Laplace transforms of the Wiener-Hopf kernel are known. In the talk, we present Monte Carlo validation of these results over a wide variety of nonclassical media types in a variety of dimensions.

Integral equations: In $\mathbb{R}^d$ the surface area $\Omega_d(r)$ of the sphere of radius r is $\Omega_d(r) = d\pi^{d/2}r^{d-1}/(\Gamma(d/2+1))$, and the isotropic scattering phase function is $P(\omega_i \to \omega_o) = 1/\Omega_d(1)$. The generalized Peierls integral equation for the scalar collision rate density $C(\mathbf{x})$ is $[2, 3]$

$$C(\mathbf{x}) = C_0(\mathbf{x}) + c\int_{\mathbb{R}^d} C(\mathbf{x}')\frac{p_c(||\mathbf{x}-\mathbf{x}'||)}{\Omega_d(||\mathbf{x}-\mathbf{x}'||)}d\mathbf{x}', \tag{1}$$

where $C_0(x)$ is the scalar rate density of initial collisions in the system. Under generalized plane-parallel symmetry in a half space, where C is uniform in all but one axis x, we find the Wiener-Hopf equation,

$$C(x) = C_0(x) + c\int_0^\infty C(x')K_C(x-x')dx' \tag{2}$$

where the collision-rate density kernel K_C and its Fourier transform $\tilde{K}_C(t)$ are

$$K_C(x) = \frac{1}{2}\int_0^1 p_c(|x|/\mu)\frac{1}{\mu}G(\mu)d\mu, \quad \tilde{K}_C(t) \equiv \int_{-\infty}^\infty K_C(x)e^{ixt}dx \tag{3}$$

using angular measure

$$G(\mu) = \frac{2(1-\mu^2)^{\frac{d-3}{2}}\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}, \quad d > 1 \tag{4}$$

with $G(\mu) = 1$ in 3D. After determining $C(x)$, the collided scalar flux $\phi_c(x)$ follows from convolution of $c\,C(x)$ with kernel K_ϕ given by Eq.(3) with $p_c(s)$ replaced by $X_c(s)$.

Green's function: We have, from Ivanov ([4], Eqs. (19) and (21)), that the double Laplace transform of the reciprocal Green's function is

$$\bar{\bar{\mathbb{G}}}(s, s_0) = \mathcal{L}_s\left[\mathcal{L}_{s_0}\left[\mathbb{G}(x, x_0)\right]\right] = \frac{H(1/s)H(1/s_0)}{s + s_0} \tag{5}$$

in terms of the H function for the given kernel K_C. H is given uniquely by [4]

$$H(z) = \exp\left(\frac{z}{\pi}\int_0^\infty \frac{1}{1 + z^2t^2}\log\left[\frac{1}{1 - c\,\tilde{K_C}(t)}\right]dt\right), \quad \operatorname{Re} z > 0 \tag{6}$$

with universal limits $H(0) = 1$ and $H(\infty) = (1 - c)^{-1/2}$. The H function satisfies [4]

$$H(1/s) = 1 + H(1/s)c\int_0^\infty \frac{H(1/s')k(s')}{s + s'}ds', \tag{7}$$

where $k(s)$ is the inverse Laplace transform of the collision rate density kernel,

$$\int_0^\infty e^{-s|x|}k(s) = K_C(x), \quad k(s) = \frac{1}{2}\int_0^1 \mathcal{L}^{-1}_{s\,u}[p_c(x)]G(u)du. \tag{8}$$

Albedo problem: In nonclassical random media, the density of initial collisions for a single photon entering along direction μ_0 is not an exponential, but rather $C_0(x) = p_u(x/\mu_0)/\mu_0$, which creates a less direct relationship between the Laplace-transformed Green's function and the diffuse reflection law. Using the inverse Laplace transforms of $p_u(s)$ and $X_c(s)$, we find the generalized law of diffuse reflection for the half space in terms of a superposition of the transformed Green's function,

$$I(0, \mu; -\mu_0) = \frac{c}{2}\int_0^\infty\int_0^\infty \mathcal{L}^{-1}_{s_0}(p_u(x))\mathcal{L}^{-1}_{s}(X_c(x))\frac{H(\mu/s)H(\mu_0/s_0)}{s\mu_0 + s_0\mu}ds\,ds_0. \tag{9}$$

Here, $I(0, \mu; -\mu_0)$ is an "azimuthally-integrated" radiance such that $I(0, \mu; -\mu_0)\mu G(\mu)$ is the rate density of energy flowing in directions with cosine μ through a surface patch of unit area. Reciprocity is achieved if and only if the free-path distribution for entering the medium $p_u(s)$ is proportional to the extinction function for leaving a collision $X_c(s)$, in agreement with a previous derivation over single-scattering paths [5]. Chandrasekhar's classical result is included above in the case that $p_u(s) = X_c(s) = e^{-s}$, with $\mathcal{L}^{-1}_{s_i}(p_u(x))$ and $\mathcal{L}^{-1}_{s_i}(X_c(x))$ given by Dirac deltas. The diffuse albedo under unidirectional illumination along cosine μ_0 is likewise generalized,

$$R(\mu_0) = \int_0^1 I(0, \mu; -\mu_0)\mu G(\mu)d\mu = 1 - \sqrt{1 - c}\int_0^\infty \mathcal{L}^{-1}_{s_i}(p_u(x))\frac{H(\mu_0/s_i)}{s_i}ds_i. \tag{10}$$

Universal properties of nonclassical transport: We also prove a number of universal properties of half space transport, which hold for any continuous free-path distribution $p_c(s)$ and any dimension $d \geq 1$, provided scattering is isotropic. When illuminated by a one-sided isotropic plane source at the boundary, the half space has universal albedo

$$R = \frac{2 - c - 2\sqrt{1 - c}}{c}. \tag{11}$$

Similarly, for grazing illumination, the total reflectance is always $R(0) = 1 - \sqrt{1-c}$. For two adjacent half spaces with differing single scattering albedos c_1, c_2 and an isotropic source at their interface, we find the total collision rate in the system

$$\int_{-\infty}^{\infty} C(x)dx = \frac{1}{\sqrt{1-c_1}\sqrt{1-c_2}}. \tag{12}$$

Markovian Binary Mixtures: For multidimensional Levermore-Pomraning random media [1] with

$$p_c(s) = \frac{w_- r_-^2 e^{-r_- s} + w_+ r_+^2 e^{-r_+ s}}{\langle\Sigma\rangle}, \qquad X_c(s) = \frac{w_- r_- e^{-r_- s} + w_+ r_+ e^{-r_+ s}}{\langle\Sigma\rangle}, \tag{13}$$

$$p_u(s) = w_- r_- e^{-r_- s} + w_+ r_+ e^{-r_+ s}, \qquad X_u(s) = w_- \mathrm{e}^{-r_- s} + w_+ \mathrm{e}^{-r_+ s}, \tag{14}$$

and constants $w_- + w_+ = 1$, $0 < r_- < r_+$, $\langle\Sigma\rangle > 0$, we find

$$\mathcal{L}_s^{-1}(p_u(x)) = w_- r_- \delta(s - r_-) + w_+ r_+ \delta(s - r_+) \tag{15}$$

in terms of Dirac delta functions δ, from which, using Eq.(9), we find the law of diffuse reflection,

$$I(0,\mu;-\mu_0) = \frac{c}{2}\left[r_+ w_+ H\left(\frac{\mu}{r_+}\right)\left(\frac{w_+ H\left(\frac{\mu_0}{r_+}\right)}{u+\mu_0} + \frac{w_- r_- H\left(\frac{\mu_0}{r_-}\right)}{r_-\mu + r_+\mu_0}\right) \right.$$
$$\left. + r_- w_- H\left(\frac{\mu}{r_-}\right)\left(\frac{w_- H\left(\frac{\mu_0}{r_-}\right)}{\mu+\mu_0} + \frac{w_+ r_+ H\left(\frac{\mu_0}{r_+}\right)}{r_-\mu_0 + r_+\mu}\right)\right]. \tag{16}$$

Combining Eqs.(15) and (10), we find the diffuse albedo under monodirectional illumination to be

$$R(\mu_0) = 1 - \sqrt{1-c}\left(w_- H\left(\frac{\mu_0}{r_-}\right) + w_+ H\left(\frac{\mu_0}{r_+}\right)\right). \tag{17}$$

The H functions for Markovian binary mixtures in Flatland are determined from

$$\tilde{K}_C(z) = \frac{1}{\langle\Sigma\rangle}\left(\frac{r_-^2 w_-}{\sqrt{r_-^2 + z^2}} + \frac{r_+^2 w_+}{\sqrt{r_+^2 + z^2}}\right), \tag{18}$$

and in 3D from

$$\tilde{K}_C(z) = \frac{1}{\langle\Sigma\rangle}\frac{w_- r_-^2 \tan^{-1}\left(\frac{z}{r_-}\right) + w_+ r_+^2 \tan^{-1}\left(\frac{z}{r_+}\right)}{z}. \tag{19}$$

Gamma random flights Gamma random flights derive from an intercollision free-path distribution that is the Laplace transform of the fractional derivative of a Dirac delta,

$$p_c(s) = \frac{\mathrm{e}^{-s} s^{a-1}}{\Gamma(a)} = \int_0^\infty \mathrm{e}^{-st} \frac{\delta^{(a-1)}(t-1)}{\Gamma(a)} dt \tag{20}$$

with parameter $a > 0$, and include classical exponential media when $a = 1$. The Fourier transform of the kernels and the Case eigenfunctions generalize, respectively, to

$$\tilde{K}_C(z) = {}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; \frac{d}{2}; -z^2\right), \quad \phi(\mu,\nu) = \frac{c}{2}\left(\frac{\nu}{\nu-\mu}\right)^a. \tag{21}$$

In 3D, we find exact analytic solutions of the albedo problem for $a \in \{2,3,4\}$. When $a = 2$, we find $C(x)$ to exactly satisfy a diffusion equation. With the escape probability from the medium given by a simple exponential of depth, we construct the perfectly zero-variance walk for escaping the medium. The diffuse albedos for monodirectional and uniform diffuse illuminations are,

$$R(\mu_0) = \frac{c\left(\sqrt{1-c}\mu_0 + 2\right)}{2\left(\sqrt{1-c}+1\right)\left(\sqrt{1-c}\mu_0+1\right)^2}, \quad R = \frac{c}{\left(\sqrt{1-c}+1\right)^2}. \tag{22}$$

The diffuse reflection law for the "diffusion-transport" half space is then the algebraic expression

$$\begin{aligned} I(0,\mu;-\mu_0) = \frac{c}{4(\mu+\mu_0)^3} &\left[\mu(\mu+\mu_0)H'(\mu)\left(\mu_0(\mu+\mu_0)H'(\mu_0) + H(\mu_0)(2\mu+\mu_0)\right)\right. \\ &\left. + H(\mu)\left(\mu_0(\mu+\mu_0)(\mu+2\mu_0)H'(\mu_0) + 2H(\mu_0)\left(\mu^2+3\mu\mu_0+\mu_0^2\right)\right)\right] \end{aligned} \tag{23}$$

with $H(\mu) = (1+\mu)/(1+\sqrt{1-c}\mu)$.

Power-law random flights For power law random media [6] with $p_u(s) = \left(\frac{a}{a+s}\right)^{a+1}$, we find, in 3D,

$$\mathcal{L}_t^{-1}\left[p_c(s)\right] = \frac{a^a e^{-at} t^{a+1}}{\Gamma(a)}, \quad k(s) = \int_0^1 \frac{a^a e^{-asu}(su)^{a+1}}{2\Gamma(a)} du = \frac{\Gamma(a+2) - \Gamma(a+2, as)}{2a^2 s \Gamma(a)}. \tag{24}$$

In the general case, the complexity of the Fourier transform of K_C presented numerical difficulty, but we were able to derive the closed form single-scattering BRDF in 3D [5],

$$f_1(\mu_i,\mu_o) = \frac{c}{4\pi} \frac{a\,{}_2F_1\left(1, a+1; 2(a+1); 1 - \frac{\mu_o}{\mu_i}\right)}{(2a+1)\mu_i}. \tag{25}$$

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